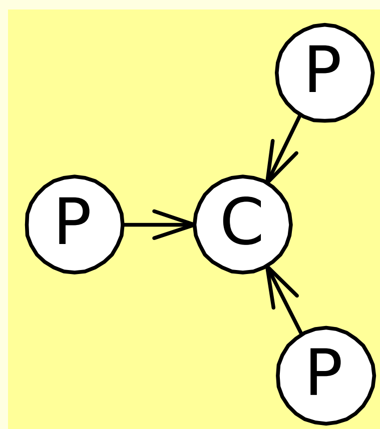
2DH Curves

Cubic

The Cubic Curve Equation

$$
\begin{aligned}
&Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \\
&+3Ex^2w + 6Fxyw + 3Gy^2w \\
&+3Hxw^2 + 3Jyw^2 \\
&+Kw^3 = 0
\end{aligned}
$$

$$
\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$



$=0$

Standard Positions

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3$$
$$=0$$

Transform to make some coefficients zero

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3$$
$$=0$$

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3$$
$$=0$$

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3$$
$$=0$$

My Favorite Standard Position

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$

$$+ 3Ex^2w + 6Fxyw + 3Gy^2w$$

$$+ 3Hxw^2 + 3Jyw^2$$

$$+ Kw^3$$

$$= 0$$

$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

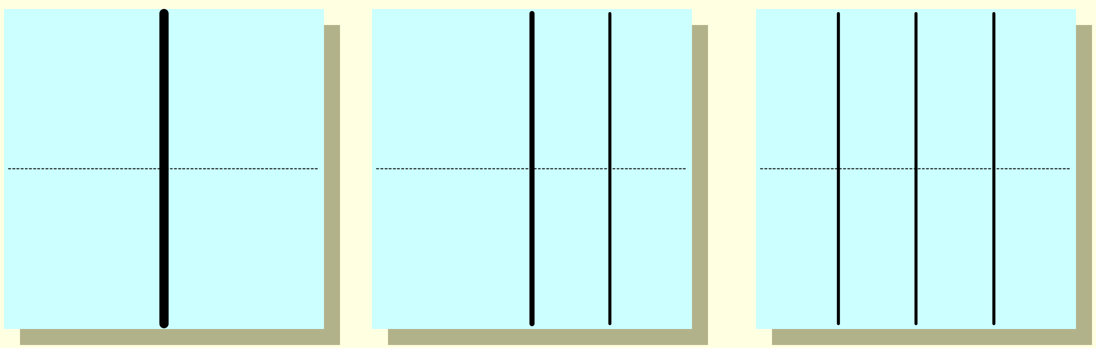
The Catalog – Reducible Cubics

$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

$$G = 0$$

ß

$$0 = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$



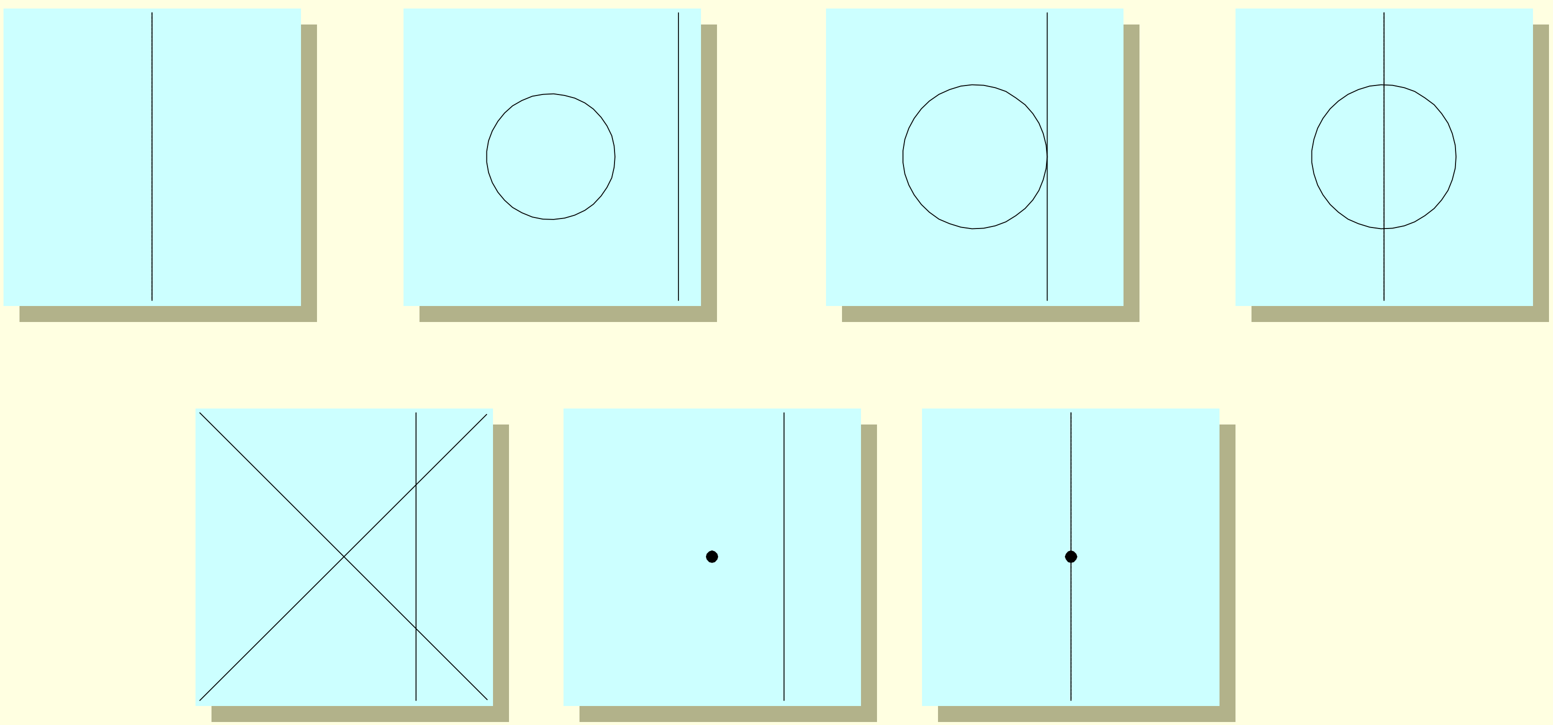
The Catalog – Reducible Cubics

$$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

$$A = 0$$

ß

$$0 = \left(3Ex^2 + 3Hxw + Kw^2 - 3Gy^2\right)w$$



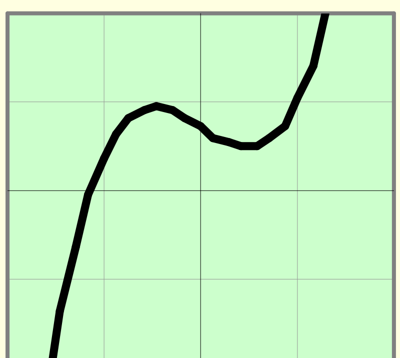
The Catalog

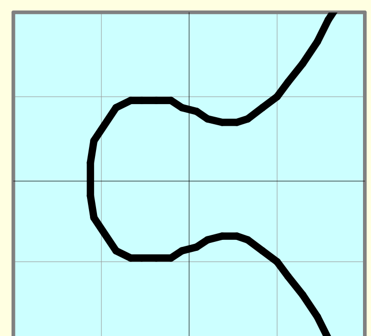
$G \neq 0$ $$-3Gy^2w = Ax^3 + 3Ex^2w + 3Hxw^2 + Kw^3$$

$A \neq 0$ $$\Downarrow$$

$$y^2w = x^3 + 3Hxw^2 + Kw^3$$

$$Y = \sqrt{X^3 + cX + d}$$





Not A Two Parameter Class

$$Y = \sqrt{X^3 + cX + d}$$

Scale in X and Y

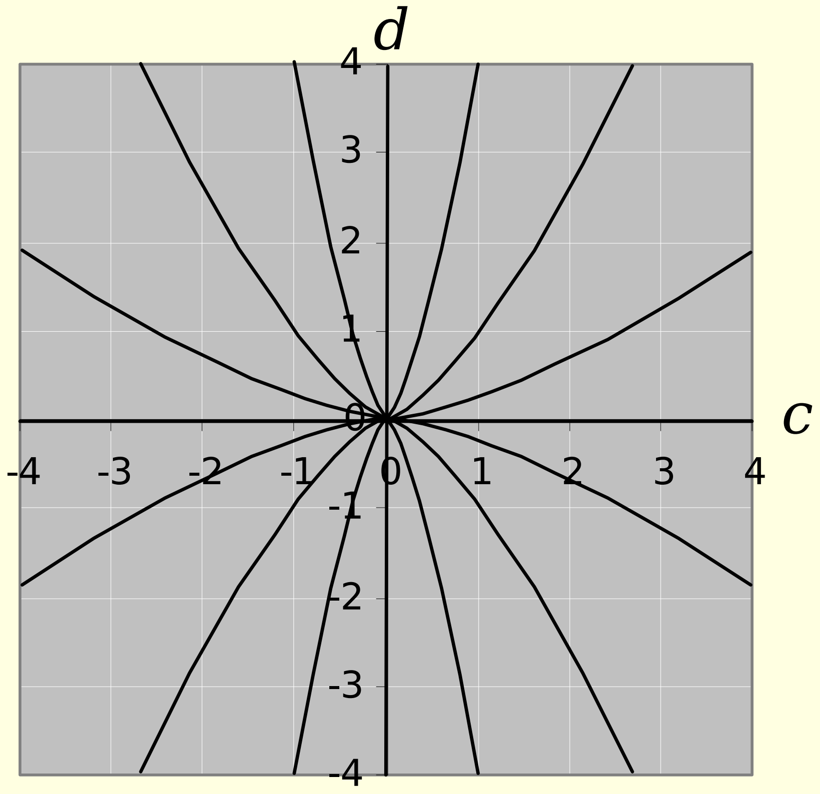
$$sY = \sqrt{\left(\sqrt[3]{s^2}X\right)^3 + c\left(\sqrt[3]{s^2}X\right) + d}$$

$$Y = \sqrt{X^3 + \hat{c}X + \hat{d}}$$

$$\hat{c} = cs^{-4/3}, \hat{d} = ds^{-2}$$

$$\frac{c^3}{d^2} = \frac{\hat{c}^3}{\hat{d}^2} = \text{constant}$$

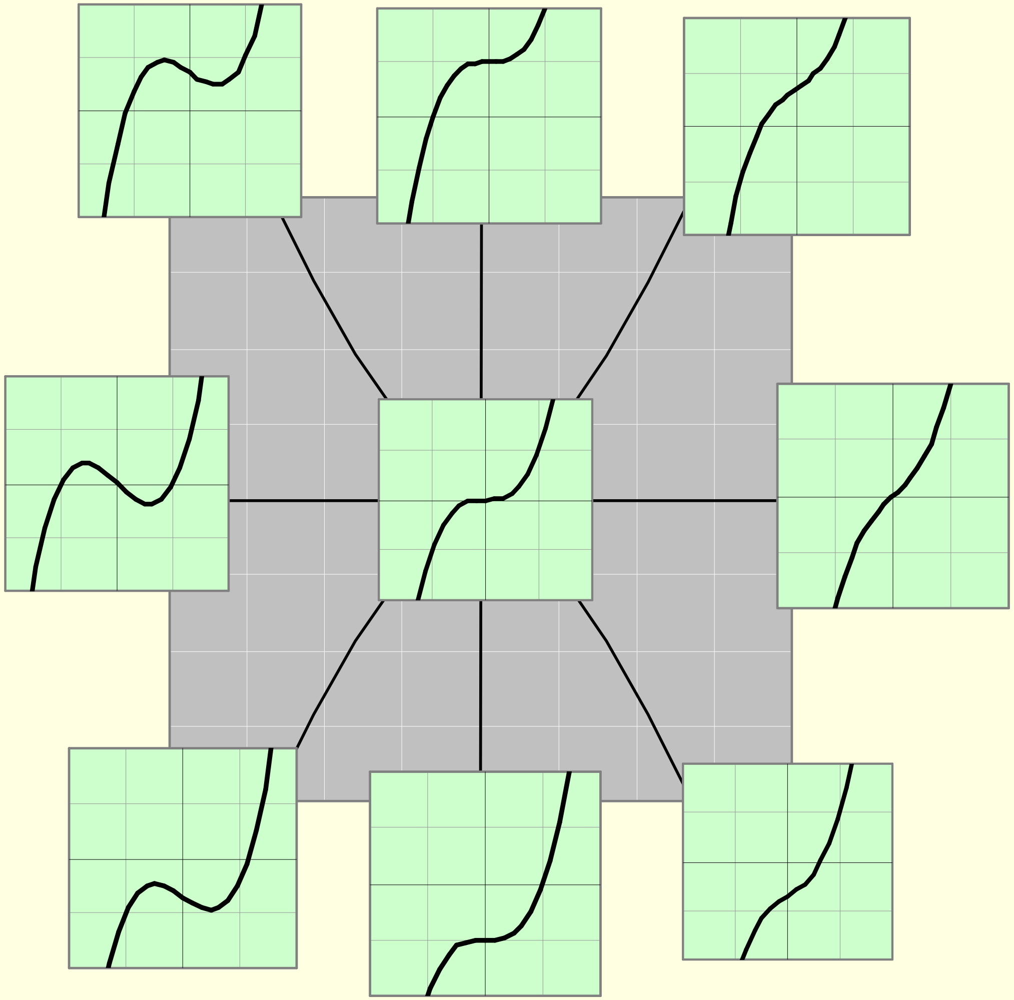
Space of Irreducible Cubics



$$\frac{c^3}{d^2} = \mathrm{constant}$$

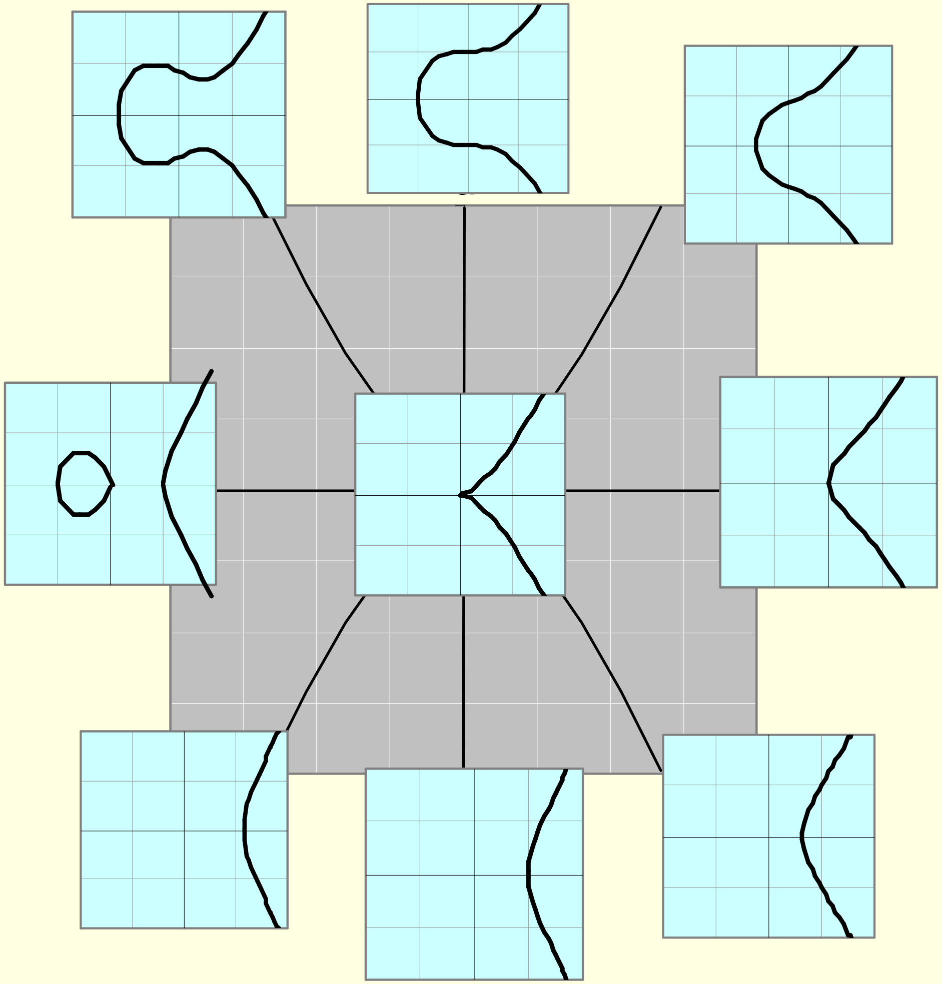
Plot Y squared

$$Y^2 = X^3 + cX + d$$



Samples of Irreducible Cubics

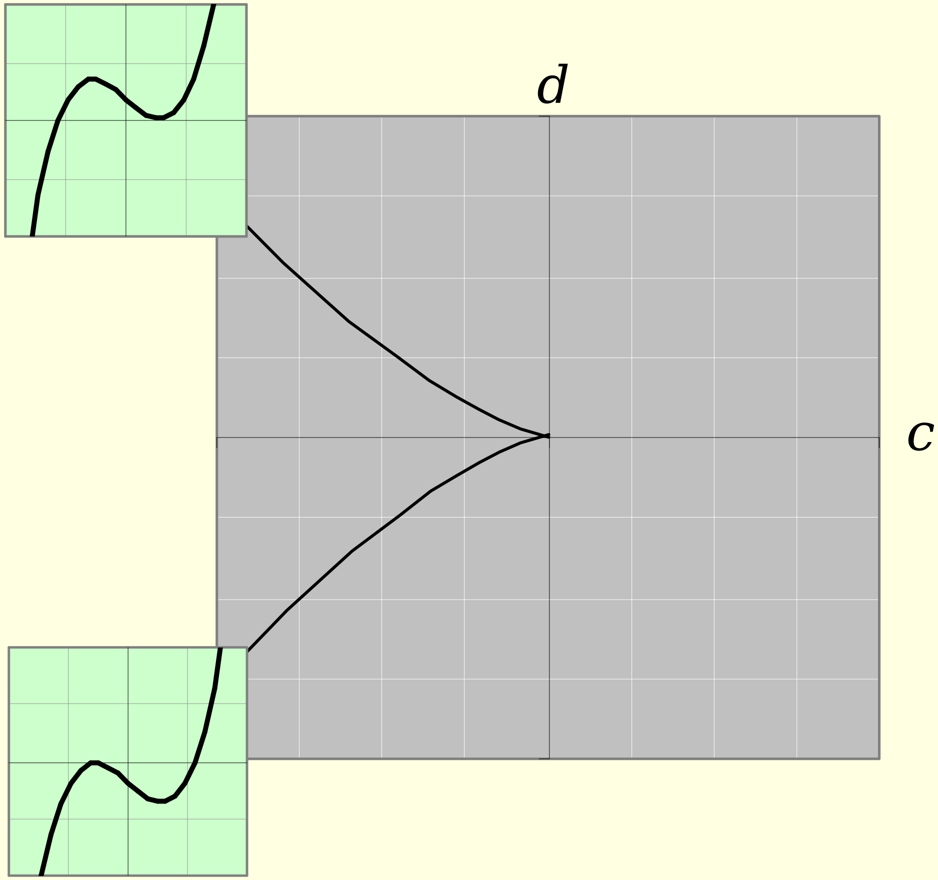
$$Y = \sqrt{X^3 + cX + d}$$



Particularly Interesting Cases

$$Y^2 = X^3 - 3X + 2$$

$$= (X+2)(X-1)^2$$

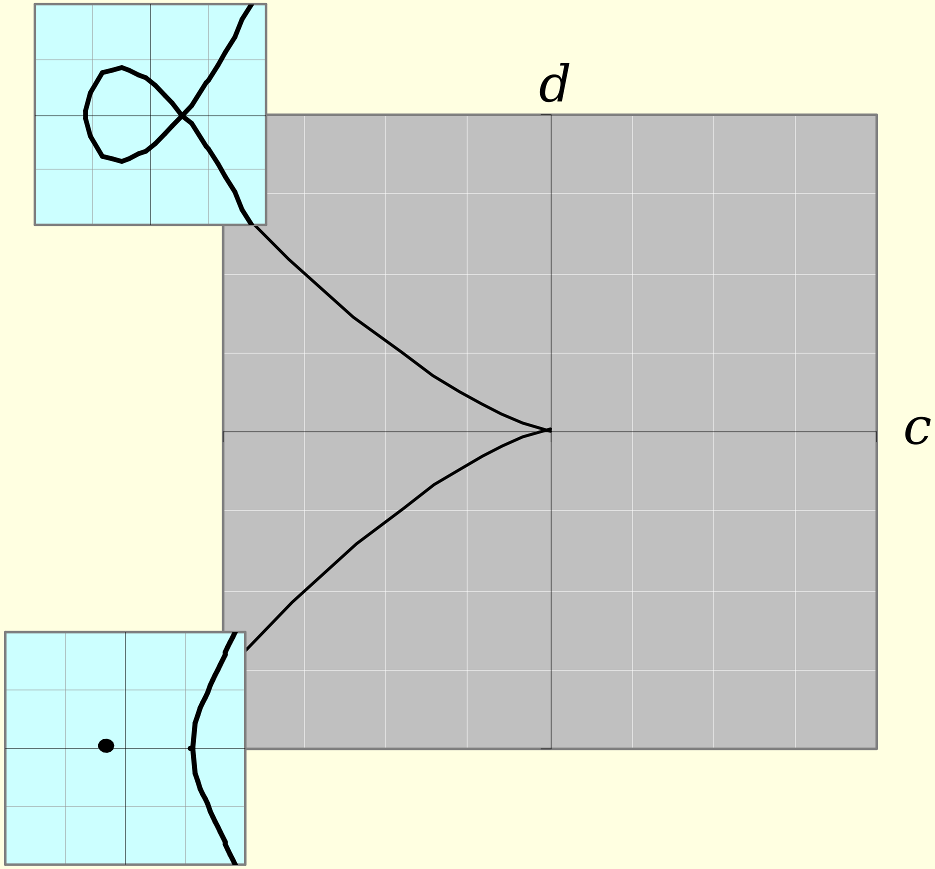


$$Y^2 = X^3 - 3X - 2$$

$$= (X-2)(X+1)^2$$

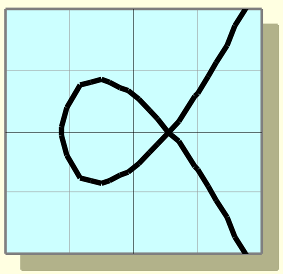
Acnode and Crunode

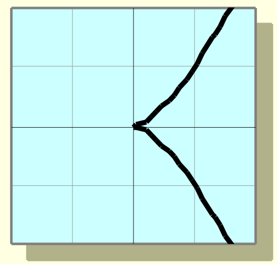
$$Y^2 = X^3 - 3X + 2$$
$$= (X+2)(X-1)^2$$

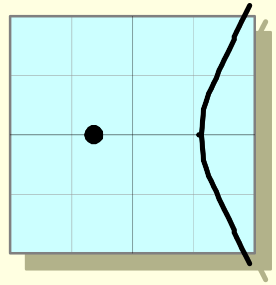


$$Y^2 = X^3 - 3X - 2$$
$$= (X-2)(X+1)^2$$

Irreducible Cubic Curves



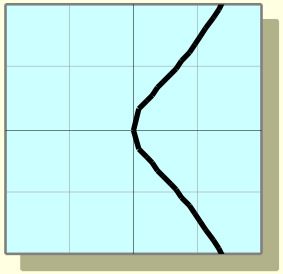


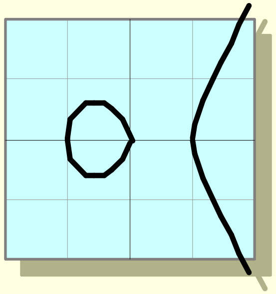


$$x^3 - 3xw^2 + 2w^3 - y^2w = 0$$

$$0 = x^3 - y^2w$$

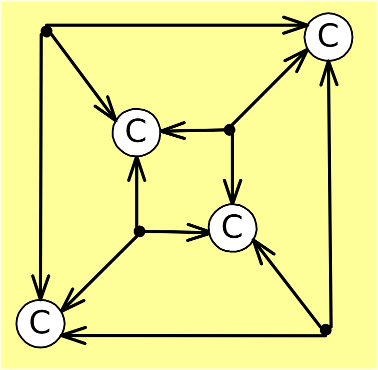
$$x^3 - 3xw^2 - 2w^3 - y^2w = 0$$



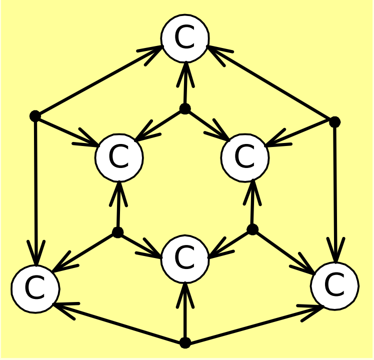


$$y^2w = x^3 + cxw^2 + dw^3$$

Invariants



$$I_{cube} = 24G^2\left(E^2 - AH\right)$$



$$I_{hexagon} = 24G^3\left(A\left(EH - AK\right) + 2E\left(AH - E^2\right)\right)$$

$$\mathbf{D} = 16A^3G^6\left(A^3K^2 + 4H^3\right)$$

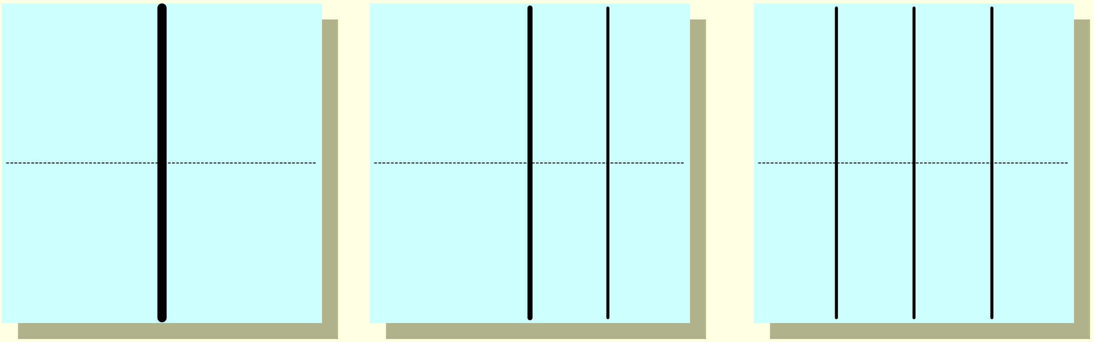
Doubly Reducible

$$I_{cube} = 0$$

$$G = 0$$

$$I_{hexagon} = 0$$

$$\mathbf{D} = 0$$

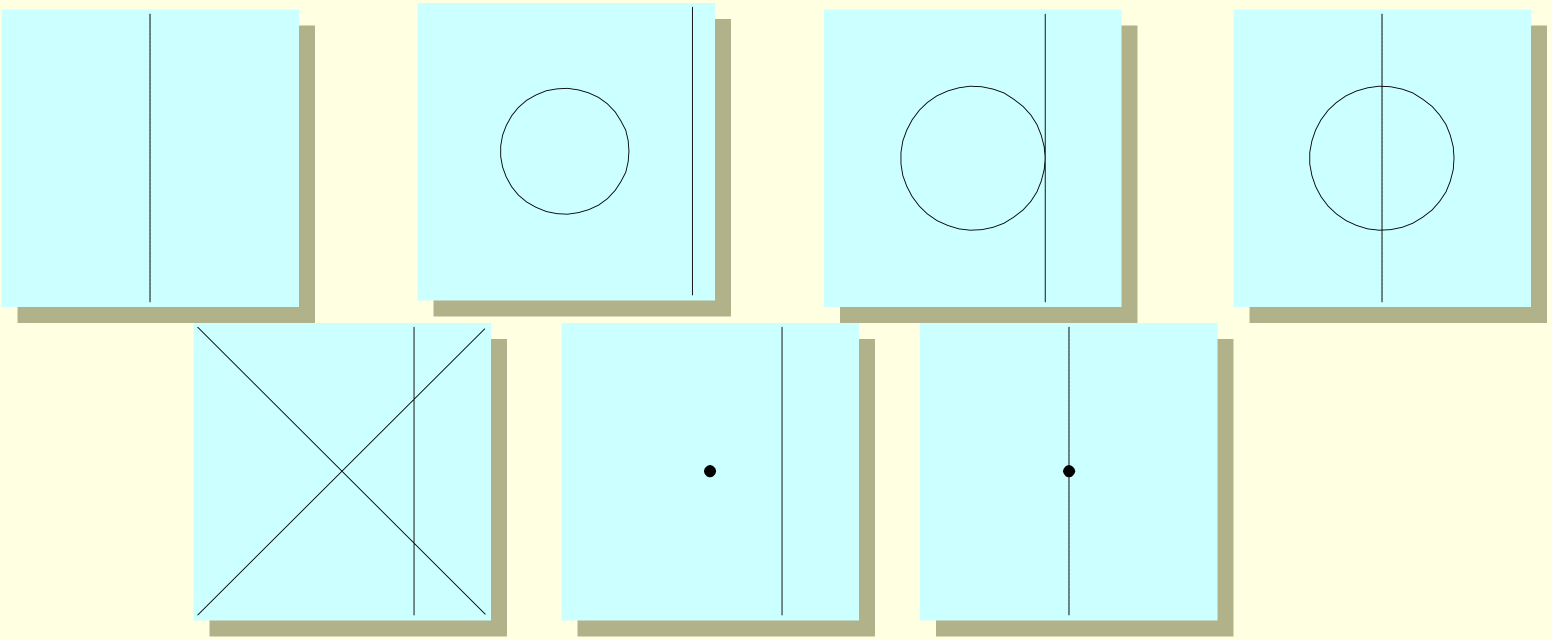


Reducible

$$I_{cube} = 24G^2E^2$$

$$A = 0 \qquad I_{hexagon} = -48G^3E^3$$

$$\mathbf{D} = 0$$

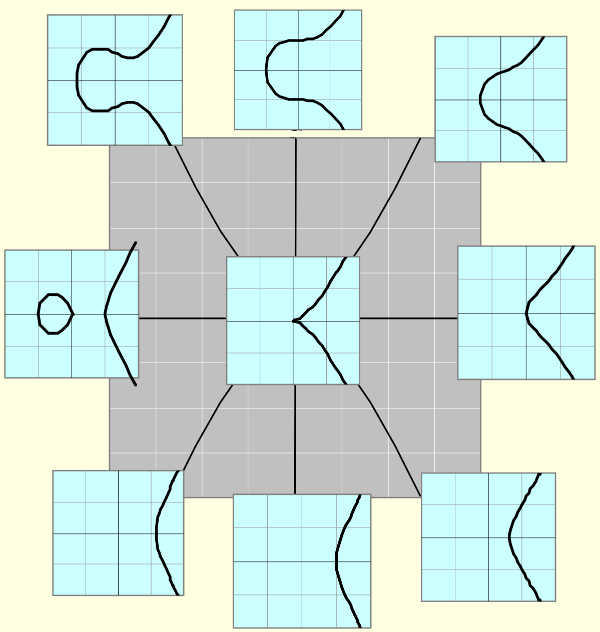


Irreducible

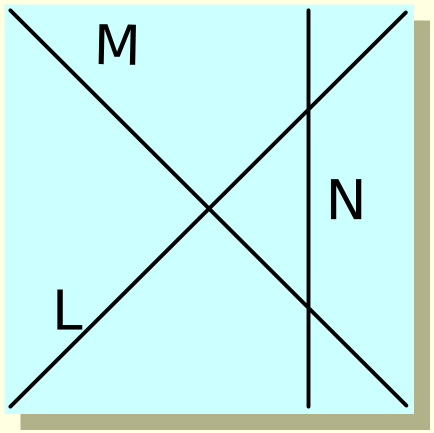
$$I_{cube} = -24H$$

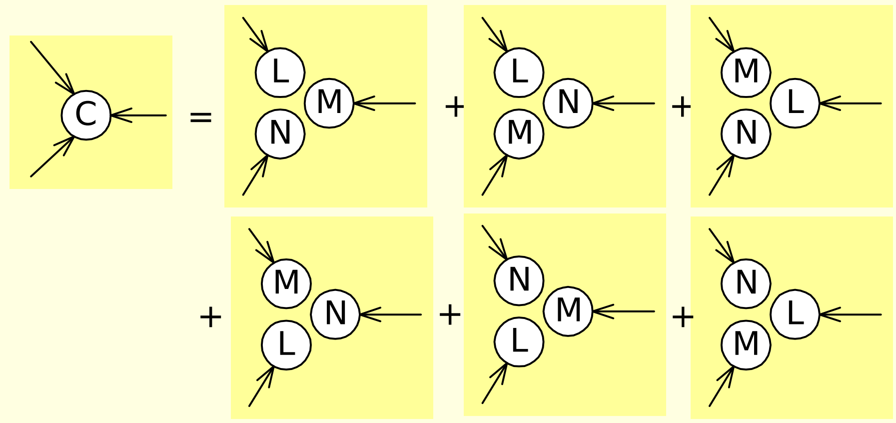
$$G = 1, A = 1, E = 0 \Rightarrow \quad I_{hexagon} = -24K$$

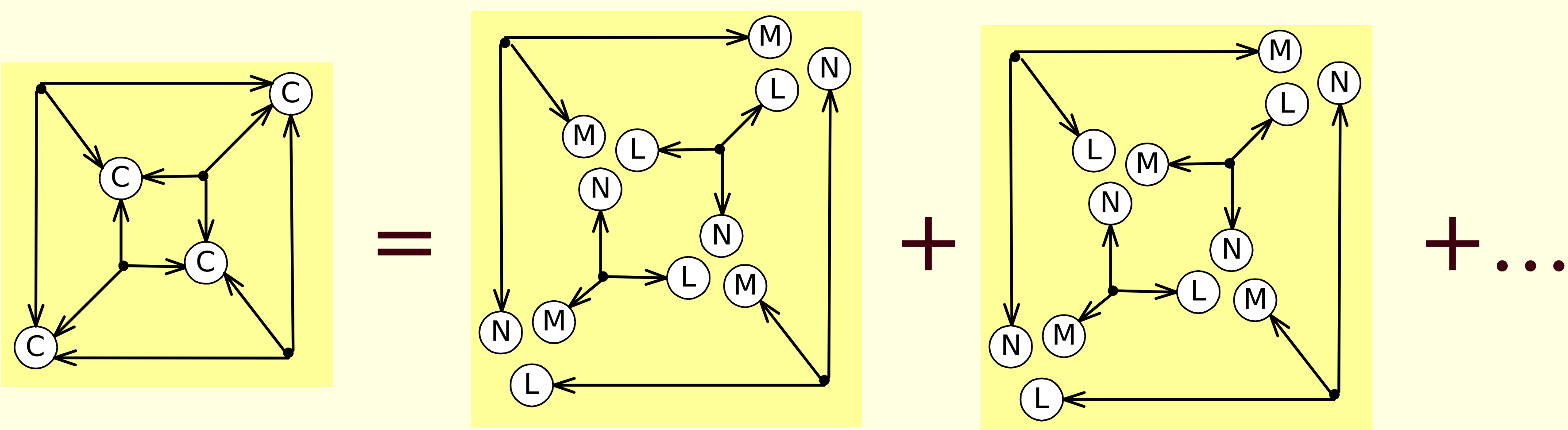
$$\Delta = 16\left(K^2 + 4H^3\right)$$



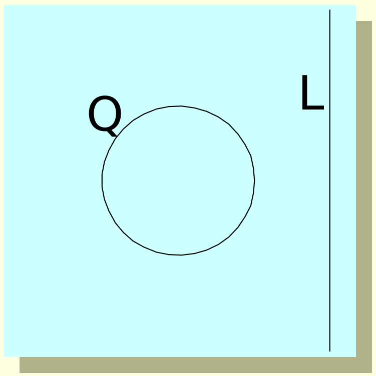
SubAtomic Cubics - Reducible

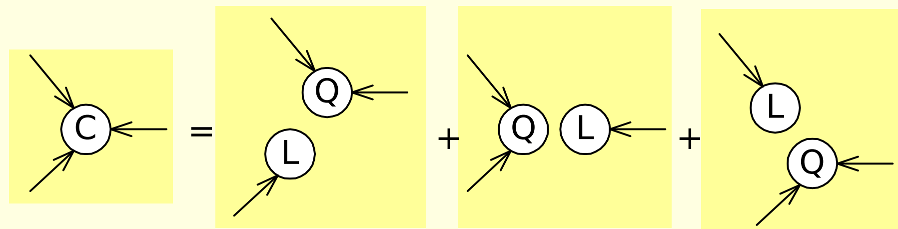


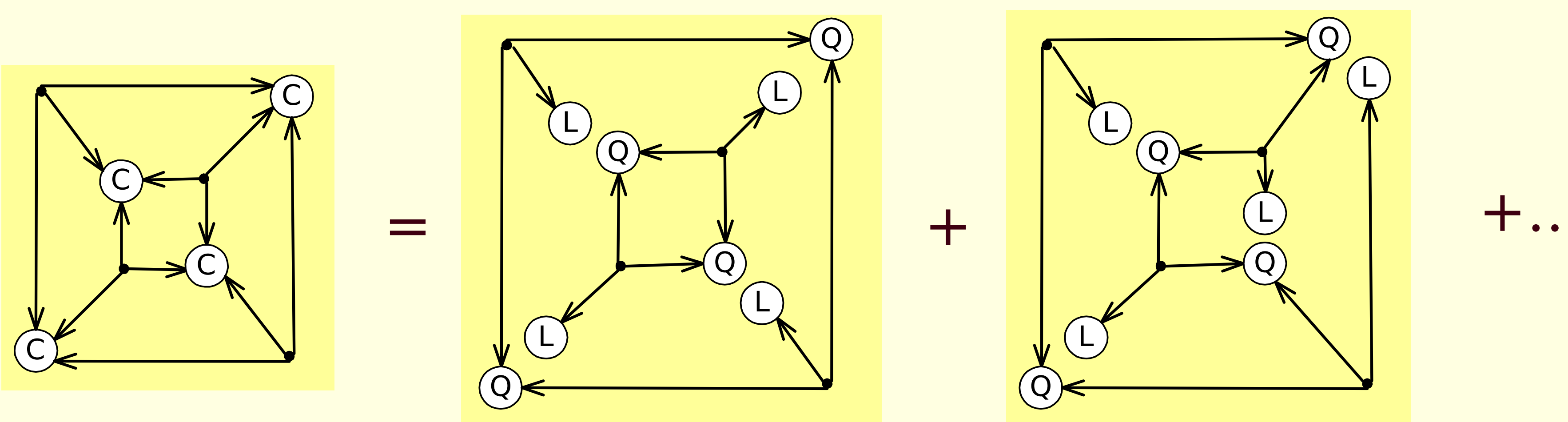




SubAtomic Cubics - Reducible

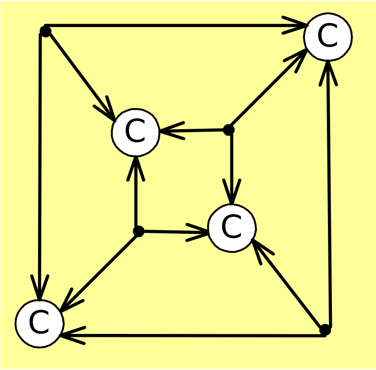


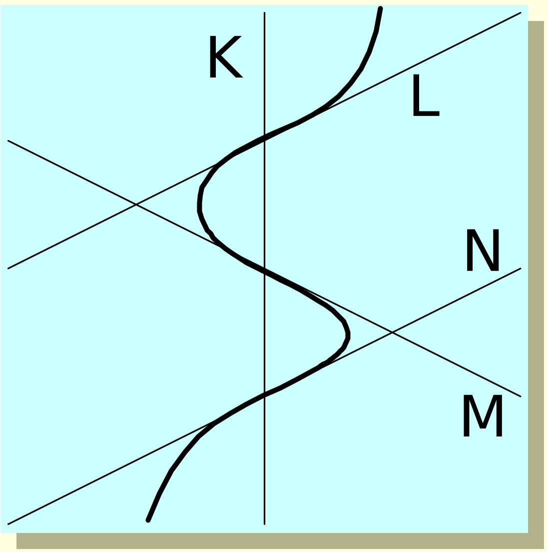


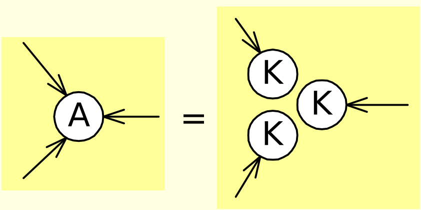


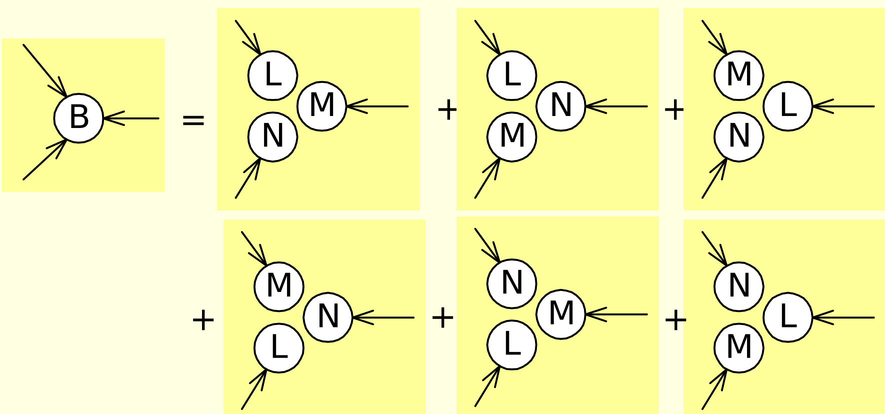
SubAtomic Cubics - Irreducible

$$\mathbf{C} = a\mathbf{A} + b\mathbf{B}$$



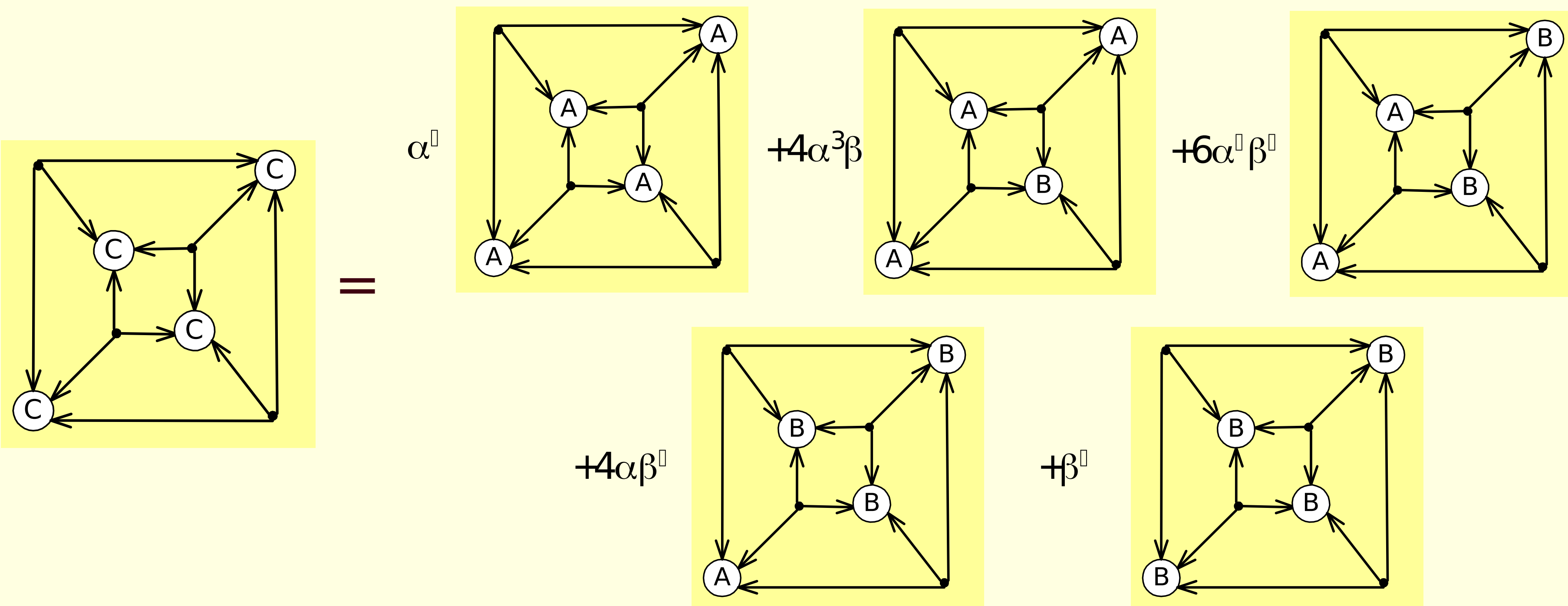




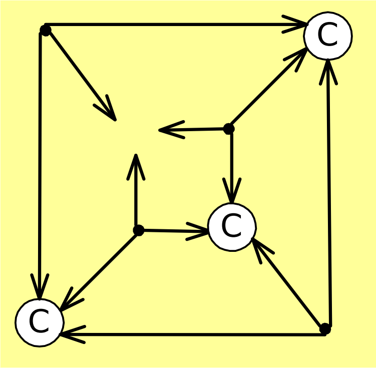


SubAtomic Cubics - Irreducible

$$\mathbf{C} = a\,\mathbf{A} + b\,\mathbf{B}$$



Caylean



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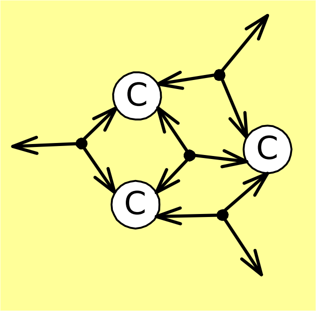
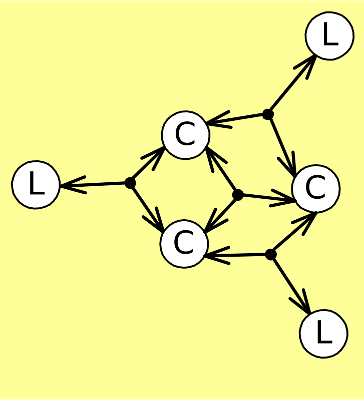




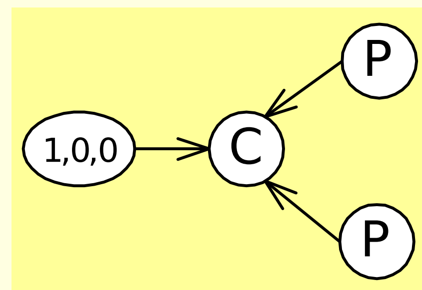
= 0 Means what?

First Derivatives

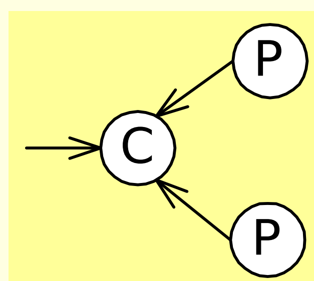
$$f(x,y,w) = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 + 3Ex^2w + 6Fxyw + 3Gy^2w + 3Hxw^2 + 3Jyw^2 + Kw^3$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \left\{ \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} = 0$$

$$\frac{\partial f}{\partial x} = f_x = 3Ax^2 + 6Bxy + 3Cy^2 + 6Exw + 6Fyw + 3Hw^2$$



$$\begin{bmatrix} f_x \\ f_y \\ f_w \end{bmatrix} = 3$$



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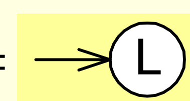


Second Derivatives

$$\begin{bmatrix} x & y & w \end{bmatrix} \left(\begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} x + \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} y + \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} w \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

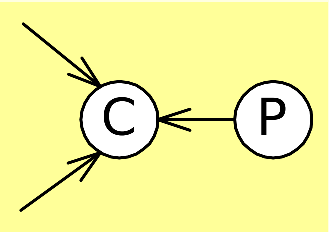
$$\frac{\partial f}{\partial x} = f_x = 3Ax^2 + 6Bxy + 3Cy^2 + 6Exw + 6Fyw + 3Hw^2$$

$$f_{xx} = 6Ax + 6By + 6Ew$$

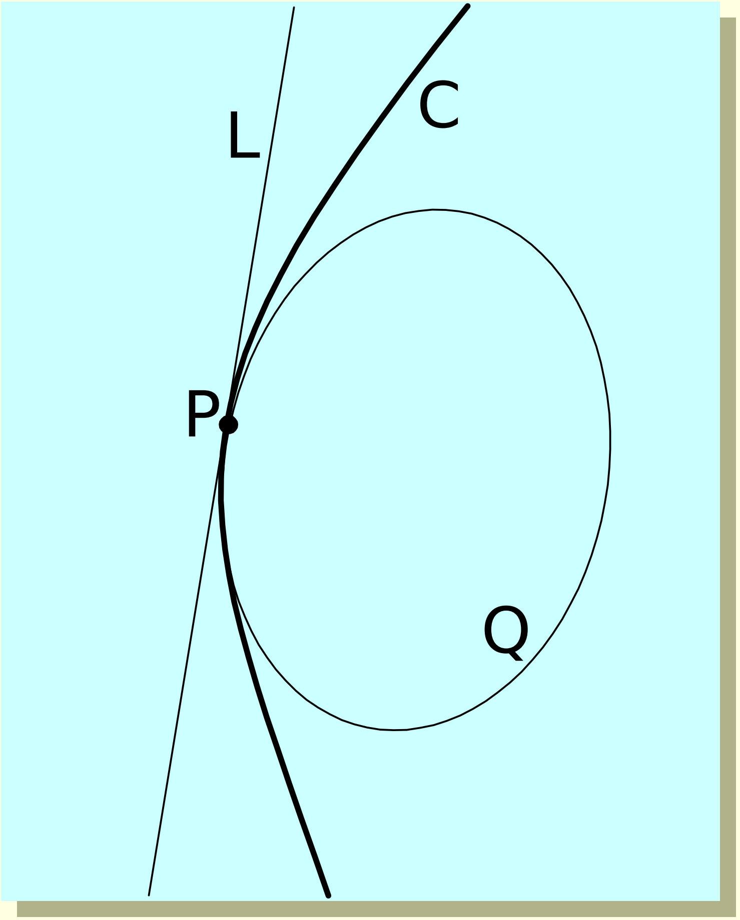
$$f_{xy} = 6Bx + 6Cy + 6Fw$$

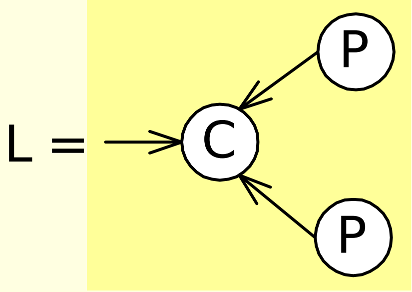
$$f_{xw} = 6Ex + 6Fy + 6Hw^2$$

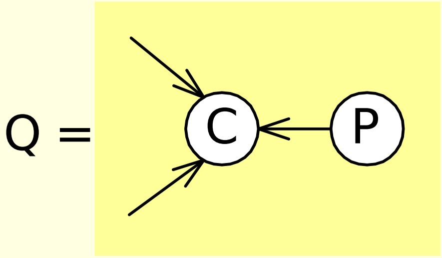
$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{xy} & f_{yy} & f_{yw} \\ f_{xw} & f_{yw} & f_{ww} \end{bmatrix} = 6$$



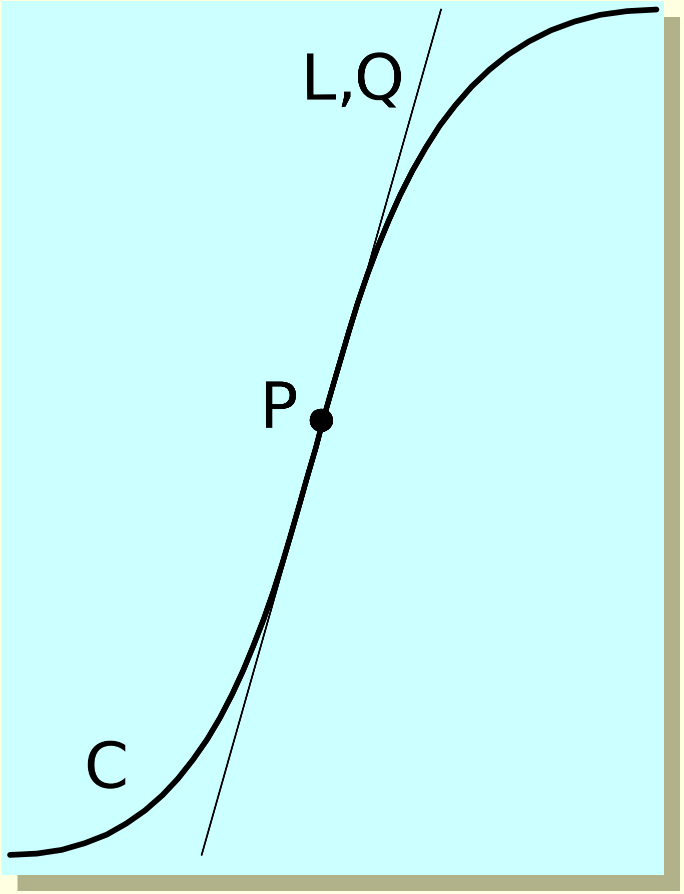
Typical Points

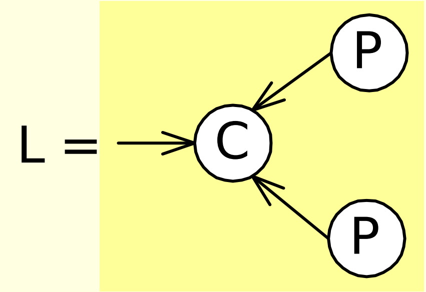


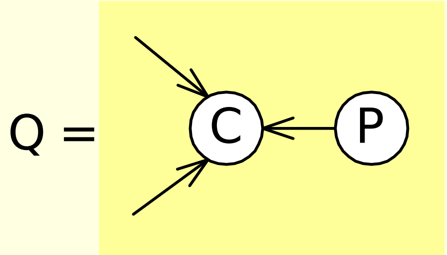




Inflection Points



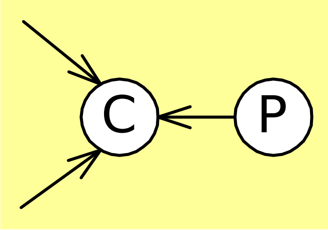




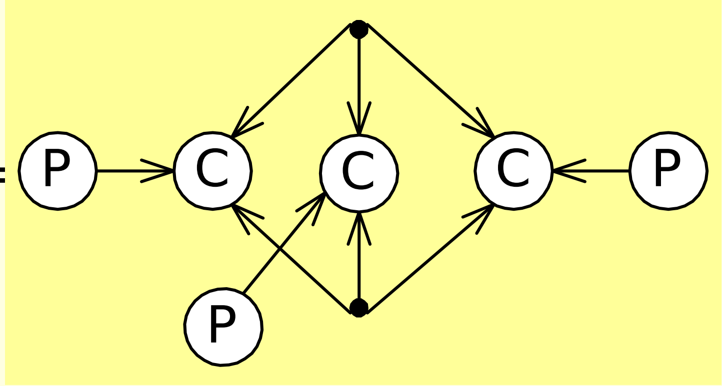
$$\det \mathbf{Q} = 0$$

Hessian

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{xy} & f_{yy} & f_{yw} \\ f_{xw} & f_{yw} & f_{ww} \end{vmatrix} = 6$$

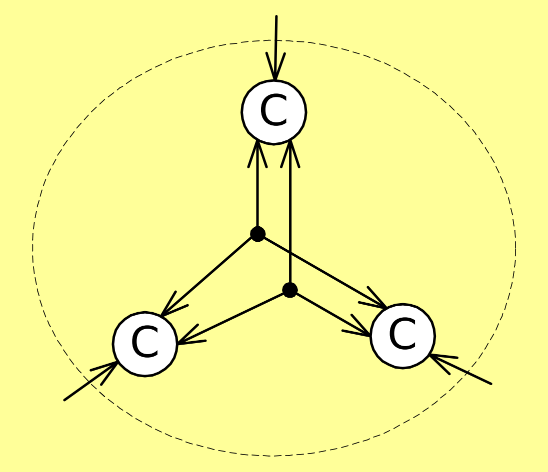


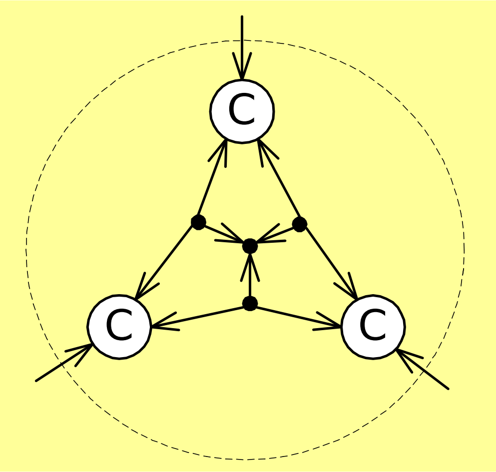
$$\mathbf{H}(x, y, w) = \det \begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{xy} & f_{yy} & f_{yw} \\ f_{xw} & f_{yw} & f_{ww} \end{bmatrix} =$$



$= 0$

Hessian Diagram Forms





Note: Hessian transforms with original curve

Hessian of Hessian